

Classification of Blow-ups and Free Boundaries of Solutions to Unstable Free Boundary Problems

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Abstract

In general, solutions u to

$$\Delta u(\mathbf{x}) = f(\mathbf{x})\chi_{\{u>\psi\}}$$

are not $C^{1,1}$, even for f smooth and $\psi(\mathbf{x}) \equiv 0$. Points around which u is not $C^{1,1}$ are called *singular points*, and the set of all such points, the *singular set*. In this article we analyze blow-ups, the free boundary $\partial\{u > \psi\}$, and the singular set close to singular points $\mathbf{x}^0 = (x^0, y^0, z^0)$ in \mathbb{R}^3 . We show that blow-ups of the form

$$\lim_{j \rightarrow \infty} \frac{u(r_j \cdot + \mathbf{x}^0)}{\|u\|_{L^\infty(B_{r_j}(\mathbf{x}^0))}},$$

$r_j \rightarrow 0^+$ are unique, the free boundary $\partial\{u > \psi\}$ is up to rotations close to the surfaces $(x - x^0)^2 + (y - y^0)^2 = 2(z - z^0)^2$ or $(x - x^0)^2 = (z - z^0)^2$, and that singular points are either isolated or contained in a C^1 curve. The methods of the proofs are based on projecting the solutions u on the space of harmonic two-homogeneous polynomials.

1 Introduction and Main Results

The obstacle problem

$$\Delta u = \chi_{\{u>0\}}, \tag{1}$$

$u \geq 0$, has been completely solved for quite some time (here χ_Ω denotes the indicator function which is equal to one on Ω and zero outside) in the sense that questions concerning existence, uniqueness, stability and regularity of solutions as well as the free boundary have been well answered [Fre72, Caf77]. However, its evil twin

$$\Delta u = -\chi_{\{u>0\}}, \tag{2}$$

also called the *unstable obstacle problem* has not until recently received some attention. The author can see at least two reasons for this: one is that it is quite new from a modelling perspective, and another reason is that techniques that can handle the obstacle problem turn out to fail for the unstable obstacle problem. Typical models that, after simplification such as looking at the stationary case, yield (2) are those of solid combustion and composite membranes. We refer to [MW07] and the references therein for a more complete description of the origin of (2).

One qualitative difference between (1) and (2) is uniqueness of solutions, and can already be illustrated in one dimension by noting that $u = \frac{x}{2}(1-x)$ and $u = 0$ both solve $u'' = -\chi_{\{u>0\}}$ in $(0,1)$ with $u(0) = u(1) = 0$. Another important distinction is the difference in regularity: it is well-known in the free boundary community that even though the right hand side in (1) is only bounded, for which $C^{1,\alpha}$ holds in general (and not better) where $0 < \alpha < 1$, one can show that solutions are $C^{1,1}$. On the other hand it has been proven in [AW06, ASW12] that there exist solutions in two dimensions and higher to (2) that have unbounded $C^{1,1}$ norm. Note however that the term “obstacle” in unstable obstacle problem is somewhat misleading since the solution is allowed to both lie below and above the “obstacle” function (which is zero) unlike the classical obstacle problem where $u \geq 0$ is imposed.

Perhaps more interesting than regularity of solutions is the regularity of the free boundary $\partial\{u > 0\}$. For the obstacle problem, the free boundary is analytic around “thick” points \mathbf{x}^0 , where blow-ups, i.e., limits

$$v(\mathbf{x}) := \lim_{r \rightarrow \infty} \frac{u(r\mathbf{x} + \mathbf{x}^0)}{r^2}$$

are halfspace solutions of the form $\frac{1}{2}(\mathbf{x} \cdot \mathbf{e}_{\mathbf{x}^0})_+^2$ for directions $\mathbf{e}_{\mathbf{x}^0} \in \partial B_1$, while it is not so easy to classify around “thin” a.k.a. singular points where blow-ups are two-homogeneous polynomials. In two dimensions it can be a cusp [Sch77], and in general the set of singular points can be included in a union of C^1 manifolds.

For the unstable obstacle problem, the singular points are analysed in [ASW10, ASW12]. We would also like to warn the reader that the term “singular” for (1) means points \mathbf{x}^0 for which blow-ups are two-homogeneous polynomials while for (2) it is used for points \mathbf{x}^0 such that $u \notin C^{1,1}(B_r(\mathbf{x}^0))$ for any $r > 0$. Because of this lack of regularity, limits of $\frac{u(r\mathbf{x} + \mathbf{x}^0)}{\|u\|_{L^\infty(B_r(\mathbf{x}^0))}}$ are considered (since $|\frac{u(r\mathbf{x} + \mathbf{x}^0)}{r^2}|$ diverges as $r \rightarrow 0^+$) and can via the Weiss monotonicity formula be shown to be a harmonic two-homogeneous polynomial [MW07].

This article will answer whether the results in [ASW12] can be extended when perturbing the right hand side both with respect to the coefficient in front of the indicator function, and the level set. In other words we answer what the blow-up limits at singular points (as used for the unstable obstacle problem), the singular set, and free boundary $\partial\{u > \psi\}$ of solutions u to

$$\Delta u(\mathbf{x}) = f(\mathbf{x})\chi_{\{u>\psi\}} \tag{3}$$

look like locally, given that f is Dini-continuous, $\psi \in C^{1,\alpha}(B_1)$, and $|\frac{\psi(r\mathbf{x} + \mathbf{x}^0)}{r^2}|$

is bounded uniformly with respect to r for singular points \mathbf{x}^0 . More precisely, we show that given a bounded solution u to (3) in $B_1 \subset \mathbb{R}^3$,

- (i) blow-ups of u of the form $\lim_{j \rightarrow \infty} \frac{u(r_j \mathbf{x} + \mathbf{x}^0)}{\|u\|_{L^\infty(B_{r_j}(\mathbf{x}^0))}}$ at singular points $\mathbf{x}^0 = (x^0, y^0, z^0)$ are unique and, up to a rotation, of the form

$$\pm \left(\frac{(x - x^0)^2 + (y - y^0)^2}{2} - (z - z^0)^2 \right),$$

or $(x - x^0)^2 - (z - z^0)^2$,

cf. Theorem 5.1, 5.2 and 5.4.

- (ii) if the blow-up at a singular point \mathbf{x}^0 is of the form $\frac{(x - x^0)^2 + (y - y^0)^2}{2} - (z - z^0)^2$ or $-\left(\frac{(x - x^0)^2 + (y - y^0)^2}{2} - (z - z^0)^2\right)$, then the free boundary $\partial\{u > \psi\}$ looks locally like the cone

$$\frac{(x - x^0)^2 + (y - y^0)^2}{2} = (z - z^0)^2$$

up to a C^1 perturbation, cf. Theorem 5.3. Furthermore, such a singular point is isolated, cf. Theorem 5.5.

- (iii) if the blow-up at a singular point \mathbf{x}^0 is of the form $(x - x^0)^2 - (z - z^0)^2$, then the set of singular points is locally contained in a C^1 -curve, cf. Theorem 5.5. Furthermore, $\{u = \psi\} \cap K_c$ is for any $c > 0$ locally contained in two C^1 -manifolds, intersecting orthogonally, where $K_c = \{y^2 < c(x^2 + z^2)\}$, cf. Theorem 5.4.

Note that even though the main results are given above, these follow by the same proof methods as in [ASW12] with some key lemmas replaced. These lemmas are presented and proved in §4 and can be considered the substantial new contributions by the author. The problem (3) is an example of a class of problems of the type $\Delta u = f(x, u)$ where f is discontinuous in u .

Finally we remark that the analysis mostly takes place in \mathbb{R}^3 , but many of the results would be true in higher dimensions as well (following [ASW13] instead of [ASW12]).

The rest of the paper has the following structure: §2 introduces the notation, and also the assumptions that nonetheless will be repeated in the lemmas, theorems and corollaries. §3 presents some important auxiliary results not proven by the author, while the main lemmas are proven in §4 from which the results, presented in §5 follows.

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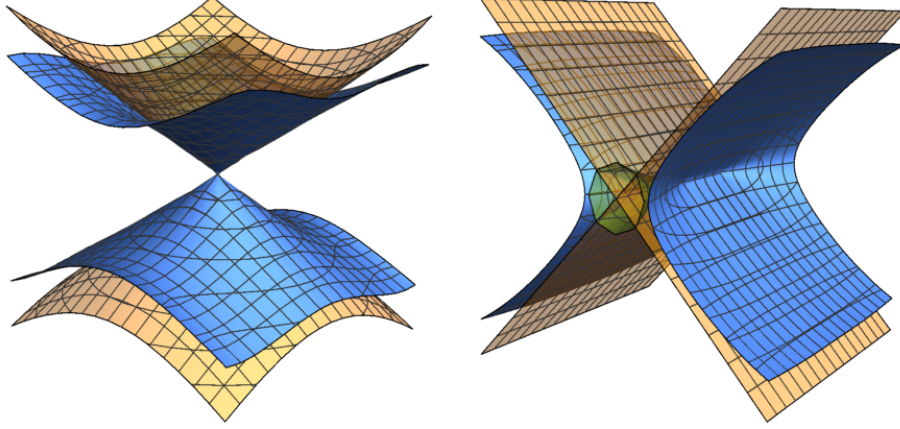


Figure 1: Illustration of (ii) and (iii); The behaviour inside the cusp is unclear.

2 Setup and Notation

As mentioned in the introduction we consider (3),

$$\Delta u(\mathbf{x}) = f(\mathbf{x})\chi_{\{u>\psi\}}$$

in the ball $B_1 \subset \mathbb{R}^n$ centered around the origin. Here, f will be assumed to be Dini continuous with modulus of continuity ω_f . The function χ_Ω is the characteristic function of the set Ω ,

$$\chi_\Omega := \begin{cases} 1, & \mathbf{x} \in \Omega, \\ 0 & \mathbf{x} \in \Omega^c. \end{cases}$$

The set $\{u > \psi\}$ is shorthand for the set $\{\mathbf{x} \in B_1 : u(\mathbf{x}) > \psi(\mathbf{x})\}$. The analysis often occurs in \mathbb{R}^3 , for which points are denoted by $\mathbf{x} = (x, y, z)$.

It will also be convenient to introduce the notation

$$h_{r,\mathbf{x}^0}(\mathbf{x}) := \frac{h(r\mathbf{x} + \mathbf{x}^0)}{r^2}, \quad (4)$$

and it will be assumed throughout that $|\psi_r| \leq C$ for some universal constant $C > 0$, the reason for which will be apparent in the proof of Lemma 4.1 and Lemma 4.2. A *singular point* \mathbf{x}^0 of a solution u of (3) is a point such that, for any $r > 0$, $u \notin C^{1,1}(B_r(\mathbf{x}^0))$, and the set of all such points, the *singular set*, is denoted by S^u . Because of the translation invariance of the Laplacian, we often fix $\mathbf{x}^0 = 0$, and let $u_r := u_{r,0}$ to decrease the notational burden. A consequence of this definition is that functions, other than polynomials, depending on a parameter differently than in (4) will have the dependence written as a superscript rather than a subscript.

The analysis will be done in the interior, hence no boundary conditions will be given. The solution will however be assumed globally bounded, say by $M > 0$. An important tool will be the following operator $\Pi : W^{2,2}(B_1) \times (0, s) \times B_{1-s} \rightarrow \mathbb{HP}_2$, where \mathbb{HP}_2 is the space of homogeneous polynomials of degree two,

$$\begin{aligned} & \frac{1}{|B_r(\mathbf{x}^0)|} \int_{B_r(\mathbf{x}^0)} \left| \frac{D^2 u(r\mathbf{y})}{r^2} - D^2 \Pi(u, r, \mathbf{x}^0) \right|^2 d\mathbf{y} \\ &= \inf_{p \in \mathbb{HP}_2} \frac{1}{|B_r(\mathbf{x}^0)|} \int_{B_r(\mathbf{x}^0)} \left| \frac{D^2 u(r\mathbf{y})}{r^2} - D^2 p \right|^2 d\mathbf{y}. \end{aligned} \quad (5)$$

Here $|B_r(\mathbf{x}^0)|$ denotes the n -dimensional Lebesgue measure of $B_r(\mathbf{x}^0)$, $D^2 u$ the Hessian of u , and $W^{2,2}(B_1)$ the usual Sobolev space. Again to ease the notation, $\Pi(u, r) := \Pi(u, r, 0)$. Some properties of Π can be found in §3.

An auxiliary problem of interest is

$$\begin{cases} \Delta Z_p = -\chi_{\{p>0\}} & \text{in } \mathbb{R}^n, \\ Z_p(0) = \nabla Z_p(0) = 0, \\ \lim_{|\mathbf{x}| \rightarrow \infty} \frac{Z_p(\mathbf{x})}{|\mathbf{x}|^3} = \Pi(Z_p, 1) = 0. \end{cases} \quad (6)$$

where the function p lies in \mathbb{HP}_2 .

The coordinate system is in the analysis often rotated; we therefore introduce \mathcal{R} to be the set of all rotations in \mathbb{R}^3 .

3 Preliminaries

The original problem

$$\Delta u = -\chi_{\{u>0\}}$$

was first introduced to the free boundary community by Monneau and Weiss [MW07]. There it was correctly conjectured that there exists singular points for solutions u , i.e., points \mathbf{x}^0 such that u is not $C^{1,1}$ in a neighbourhood of \mathbf{x}^0 . This was later proved by Andersson and Weiss [AW06], and a small cascade of papers followed by Andersson, Shahgholian and Weiss [ASW10, ASW12, ASW13] where the structure of the singular set is analyzed in detail.

Many of the tools used in these papers apply equally well to our problem

$$\Delta u = f \chi_{\{u>\psi\}},$$

and are presented below, including references to the proofs. First we will recall some properties of Π introduced in the previous section, and can be found in [ASW12, Lemma 3.6]:

Lemma 3.1. *Let $\Pi(u, r, \mathbf{x}^0)$ be the two-homogeneous polynomial given by (5). Then*

- (i) Π is a projection operator, i.e., $\Pi(\Pi(u, r), r) = \Pi(u, r)$.

(ii) $\Pi(h, r) = \Pi(h, s)$ for any harmonic function h and $0 < r, s < 1$.

It is also useful to note that, with the notation introduced in §2, $\Pi(u, r) = \Pi(u_r, 1)$ by the definition of Π .

The second tool is a result concerning Fourier expansions of solutions to the Poisson equation $\Delta u = \sigma$ when the right hand side is zero-homogeneous, i.e. $\sigma(r\mathbf{x}) = \sigma(\mathbf{x})$ for all $r > 0$.

Theorem 3.2 ([KM96], Theorem 3.1). *Let u solve $\Delta u = \sigma$ for a zero-homogeneous function σ such that where p_j are j -homogeneous harmonic polynomials, and assume $u(0) = |\nabla u(0)| = \lim_{|x| \rightarrow \infty} \frac{u(\mathbf{x})}{|\mathbf{x}|^3} = 0$. Then*

$$u(\mathbf{x}) = \frac{a_2}{n+2} p_2 \ln |\mathbf{x}| + |\mathbf{x}|^2 \sum_{j \neq 2} \frac{a_j}{(n+j)(j-2)} p_j(\mathbf{x}/|\mathbf{x}|).$$

$$\sigma(\mathbf{x}) = \sum_{j=0}^{\infty} a_j p_j(\mathbf{x}/|\mathbf{x}|),$$

where p_j are j -homogeneous harmonic polynomials, and assume $u(0) = |\nabla u(0)| = \lim_{|x| \rightarrow \infty} \frac{u(\mathbf{x})}{|\mathbf{x}|^3} = 0$. Then

$$u(\mathbf{x}) = \frac{a_2}{n+2} p_2 \ln |\mathbf{x}| + |\mathbf{x}|^2 \sum_{j \neq 2} \frac{a_j}{(n+j)(j-2)} p_j(\mathbf{x}/|\mathbf{x}|).$$

In [ASW12] the authors apply this result to the case $\sigma = -\chi_{\{p>0\}}$, $p \in \mathbb{HP}_2$ so that

$$Z_p(\mathbf{x}) = \frac{a_2}{n+2} p_2 \ln |\mathbf{x}| + |\mathbf{x}|^2 \sum_{j \neq 2} \frac{a_j}{(n+j)(j-2)} p_j(\mathbf{x}/|\mathbf{x}|).$$

For reasons related to the analysis, and that will be apparent in §5, we introduce a parametrization of harmonic polynomials in diagonal form with supremum norm one, $p_\delta := \pm[(1/2 + \delta)x^2 + (1/2 - \delta)y^2 - z^2]$, for $\delta \in [0, 1/2]$. Since δ is unique, we can define $\delta(u_r)$ as the δ such that $\frac{\Pi(u_r, 1)}{\|\Pi(u_r, 1)\|_{L^\infty(B_1)}}$ after rotations and relabeling of the coordinate axes are of the form p_δ . Choosing the basis $3x^2 - |x|^2$, $3x^2 - |y|^2$ and $3x^2 - |z|^2$ for the axisymmetric second order polynomials, it follows that

$$\begin{aligned} & \Pi(Z_{p_\delta}, 1/2) \\ &= \frac{(3A_x(\delta) - A(\delta))x^2 + (3A_y(\delta) - A(\delta))y^2 + (3A_z(\delta) - A(\delta))z^2}{\|3x^2 - 1\|_{L^2(\partial B_1)}}, \end{aligned} \quad (7)$$

where the coefficients A_x , A_y , A_z and A are calculated by

$$\begin{aligned} A_x(\delta) &= (-\chi_{\{p_\delta > 0\}}, x^2) := \int_{\partial B_1} -\chi_{\{p_\delta > 0\}} x^2 dS, \\ A_y(\delta) &= (-\chi_{\{p_\delta > 0\}}, y^2) := \int_{\partial B_1} -\chi_{\{p_\delta > 0\}} y^2 dS, \\ A_z(\delta) &= (-\chi_{\{p_\delta > 0\}}, z^2) := \int_{\partial B_1} -\chi_{\{p_\delta > 0\}} z^2 dS, \\ A(\delta) &= (-\chi_{\{p_\delta > 0\}}, 1) := \int_{\partial B_1} -\chi_{\{p_\delta > 0\}} dS. \end{aligned}$$

What follows is a collection of properties involving the coefficients of $\Pi(Z_{p_\delta}, 1/2)$ and an estimate of the growth of $|Cp_\delta + \Pi(Z_{p_\delta}, 1/2)|$ from below.

Lemma 3.3 ([ASW12], Lemma 4.5-6). *For $\delta \in (0, 1/2)$, η_0 and large enough C ,*

$$\begin{aligned} \sup_{B_1} |Cp_\delta + \Pi(Z_{p_\delta}, 1/2)| &\geq C + \eta_0. \\ (1 + 2\delta) \frac{3A_y(\delta) - A(\delta)}{3A_x(\delta) - A(\delta)} - 1 + 2\delta &\leq 4\delta \\ (1 + 2\delta) \frac{3A_y(\delta) - A(\delta)}{3A_x(\delta) - A(\delta)} - 1 + 2\delta &\geq 2\delta, \quad \delta \leq c_0 \\ (1 + 2\delta) \frac{3A_y(\delta) - A(\delta)}{3A_x(\delta) - A(\delta)} - 1 + 2\delta &\geq c(\beta), \quad \delta \in [\beta, 1 - \beta] \end{aligned}$$

The following useful lemma – found in [Gan01] – is of independent interest and gives an estimate of the size of sublevel sets of $|p|$ where p is a second order polynomial.

Lemma 3.4 ([Gan01], Corollary 4.1). *Let p be a second order polynomial in \mathbb{R}^n . With $\|p\|_{L^\infty(Q_1)} \leq 1$, Q_1 being the unit cube. Then*

$$|\{|p| \leq \epsilon\} \cap Q_1| \leq C(n, \alpha) \epsilon^\alpha, \quad \alpha \in (0, 1/4).$$

From the estimates given below we have uniform bounds when considering blow-ups. Heuristically they state that if u_r behaves badly at some point, then that behaviour is inherited by $\Pi(u_r, 1)$.

Lemma 3.5 ([ASW10], Lemma 5.1). *Let u solve $|\Delta u| \leq C$ in B_1 such that $\|u\|_{L^\infty(B_1)} \leq M$. If $\mathbf{x}^0 \in B_{1/2}$, $0 < r \leq 1/4$, and $u(\mathbf{x}^0) = |\nabla u(\mathbf{x}^0)| = 0$, then*

$$\|u_{r, \mathbf{x}^0} - \Pi(u_r, 1)\|_{C^{1, \alpha}(\overline{B_1})} \leq C(n, M, \alpha), \quad (8)$$

$$\|u_{r, \mathbf{x}^0} - \Pi(u_r, 1)\|_{W^{2, p}(B_1)} \leq C(n, M, p). \quad (9)$$

The proof in [ASW10, Lemma 5.1] can be applied and goes through word by word, except that $|\Delta u| \leq C$ replaces $\Delta u = -\chi_{\{u > 0\}}$.

A final remark is that the problem

$$\Delta u(x) = f(x)\chi_{\{u>0\}} + g(x)$$

can be reduced to (3) by replacing u with $u + \psi$ where ψ solves $\Delta\psi = -g$, and the analysis applies as long as ψ is $C^{1,1}$.

4 Growth of Solutions

The first lemma in this section shows that the growth of $\Pi(u_r, 1, \mathbf{x}^0)$ is logarithmic in r around singular points \mathbf{x}^0 . The analysis is performed at the origin, but is identical for any singular point \mathbf{x}^0 .

Lemma 4.1. *Let u solve (3) in $B_1 \subset \mathbb{R}^3$ for $\|u\|_{L^\infty(B_1)} \leq M$ such that $u(0) = |\nabla u(0)| = 0$, $\sup_{0 < r \leq 1/4} |\psi_r| \leq C_\psi$, and f is continuous with modulus of continuity ω_f and $f(0) = -a$, $a > 0$. Then there exist constants K_0 and $r_0 = r_0(K_0, C_\psi, f)$ such that if*

$$\|\Pi(u_r, 1)\|_{L^\infty(B_1)} \geq K_0, \quad 0 < r \leq r_0$$

then

$$\|\Pi(u_r, 2^{-j})\|_{L^\infty(B_1)} \geq \|\Pi(u_r, 1)\|_{L^\infty(B_1)} + \frac{ja\eta_0}{2} \quad (10)$$

$$\|\Pi(u_r, 2^{-j})\|_{L^\infty(B_1)} \leq \|\Pi(u_r, 1)\|_{L^\infty(B_1)} + ja\kappa_0 \quad (11)$$

and

$$\begin{aligned} \|u\|_{L^\infty(B_s)} &\geq \frac{1}{16} \left(\left(\frac{s}{r} \right)^2 \|u\|_{L^\infty(B_r)} + \eta_0 a s^2 \ln(r/s) \right), \\ \|u\|_{L^\infty(B_s)} &\leq 16 \left(\left(\frac{s}{r} \right)^2 \|u\|_{L^\infty(B_r)} + \kappa_0 a s^2 \ln(r/s) \right), \end{aligned}$$

$0 < s \leq r$. Here η_0 is the constant in Lemma 3.3 and $\kappa_0 = \kappa_0(M, \alpha)$.

Proof. Assume for the moment that $f(0) = -1$. The proof goes via contradiction: take a sequence $\{u^k\}$ of solutions to (3) bounded by M such that $\|\Pi(u_{r_k}^k, 1)\|_{L^\infty(B_1)} \geq k$ while

$$\|\Pi(u_{r_k}^k, 1/2)\|_{L^\infty(B_1)} \leq \|\Pi(u_{r_k}^k, 1)\|_{L^\infty(B_1)} + \eta_0/2. \quad (12)$$

By Lemma 3.5 $\{u_{r_k}^k - \Pi(u_{r_k}^k, 1)\}$ converges in $C_{\text{loc}}^{1,\alpha}(\mathbb{R}^n)$, up to a subsequence, to a function v as $r_k \rightarrow 0^+$. Passing again to a subsequence such that $p := \lim_k \frac{\Pi(u_{r_k}^k, 1)}{\|\Pi(u_{r_k}^k, 1)\|_{L^\infty(B_1)}} \in \mathbb{HP}_2$ and $p_k := \frac{\Pi(u_{r_k}^k, 1)}{\|\Pi(u_{r_k}^k, 1)\|_{L^\infty(B_1)}}$, we see that v solves

$$\Delta v = \lim_{k \rightarrow \infty} f(r_k \mathbf{x}) \chi_{\{u_{r_k}^k > \psi_{r_k}\}} = -\chi_{\{p > 0\}}. \quad (13)$$

Indeed,

$$u_{r_k}^k > \psi_{r_k} \Leftrightarrow \frac{u_{r_k}^k - \Pi(u_{r_k}^k, 1)}{\|\Pi(u_{r_k}^k, 1)\|_{L^\infty(B_1)}} + \frac{\Pi(u_{r_k}^k, 1)}{\|\Pi(u_{r_k}^k, 1)\|_{L^\infty(B_1)}} > \frac{\psi_{r_k}}{\|\Pi(u_{r_k}^k, 1)\|_{L^\infty(B_1)}}$$

and from the assumptions,

$$\max \left\{ \frac{|\psi_{r_k}|}{\|\Pi(u_{r_k}^k, 1)\|_{L^\infty(B_1)}}, \frac{|u_{r_k}^k - \Pi(u_{r_k}^k, 1)|}{\|\Pi(u_{r_k}^k, 1)\|_{L^\infty(B_1)}} \right\} \leq \frac{C}{k} \rightarrow 0, \quad k \rightarrow \infty,$$

from which $u_{r_k}^k > \psi_{r_k}$ is equivalent to $p > h^k$ for some function h^k tending to zero as k increases. Therefore, for any $\epsilon > 0$ and $\phi \in C_c^1(B_R)$,

$$\begin{aligned} & \lim_{k \rightarrow \infty} \left| \int_{B_R} \phi(f(r_k x) \chi_{\{h^k > 0\}} + \chi_{\{p > 0\}}) dx \right| \\ & \leq \|\phi\|_{L^\infty(B_R)} \lim_{k \rightarrow \infty} \int_{B_R} |f(r_k x) \chi_{\{u_{r_k}^k > \psi_{r_k}\}} - f(0) \chi_{\{u_{r_k}^k > \psi_{r_k}\}}| dx \\ & \quad + \|\phi\|_{L^\infty(B_R)} \lim_{k \rightarrow \infty} \int_{B_R} |\chi_{\{p > 0\}} - \chi_{\{p > h^k\}}| dx \\ & \leq \|\phi\|_{L^\infty(B_R)} (|B_R| \lim_{k \rightarrow \infty} \omega_f(r_k) + |\{p \leq \epsilon\} \cap B_R|) \\ & \leq C(R) \|\phi\|_{L^\infty(B_R)} \epsilon^\alpha, \end{aligned}$$

by Lemma 3.4. Hence (13) is proven and $v = Z_p$ by uniqueness for Z_p given in §2. Consequently

$$\lim_{j \rightarrow \infty} [\Pi(u_{r_k}^k, 1/2) - \Pi(u_{r_k}^k, 1)] = \Pi(Z_p, 1/2).$$

However, then by Lemma (3.3),

$$\begin{aligned} \|\Pi(u_{r_k}^k, 1/2)\|_{L^\infty(B_1)} & \geq \|\Pi(u_{r_k}^k, 1) + \Pi(Z_{p_k}, 1/2)\|_{L^\infty(B_1)} + o(1) \\ & \geq \|\Pi(u_{r_k}^k, 1)\|_{L^\infty(B_1)} + \eta_0 + o(1) \end{aligned}$$

which is a contradiction to (12), so (10) holds for $j = 1$ and $a = 1$. This can be iterated, recalling that $\Pi(u_r, s) = \Pi(u_{sr}, 1)$,

$$\begin{aligned} & \|\Pi(u_r, 2^{-j})\|_{L^\infty(B_1)} \\ & = \|\Pi(u_{2^{-j+1}r}, 1/2)\|_{L^\infty(B_1)} \geq \|\Pi(u_{2^{-j+1}r}, 1)\|_{L^\infty(B_1)} + \eta_0/2 \\ & = \|\Pi(u_{2^{-j+2}r}, 1/2)\|_{L^\infty(B_1)} + \eta_0/2 \geq \|\Pi(u_{2^{-j+2}r}, 1)\|_{L^\infty(B_1)} + 2\eta_0/2 \\ & \geq \dots \geq \|\Pi(u_r, 1)\|_{L^\infty(B_1)} + j\eta_0/2, \end{aligned}$$

which is the first inequality in (10).

The bound from above follows from the mean value property of harmonic functions,

$$\begin{aligned}
& |\Pi(u_r, 1/2) - \Pi(u_r, 1)|^2 \\
&= \left| \int_{B_{r/2}} \Pi(u_r, 1/2) - \Pi(u_r, 1) \, d\mathbf{x} \right|^2 \\
&\leq 2 \int_{B_{r/2}} |\Pi(u_r, 1/2) - D^2 u_r|^2 \, d\mathbf{x} + 2 \int_{B_{r/2}} |D^2 u_r - \Pi(u_r, 1)|^2 \, d\mathbf{x} \\
&\leq 2C + 2^{n+1} \int_{B_r} |D^2 u - \Pi(u_r, 1)|^2 \, d\mathbf{x} \leq 2^{n+2} C
\end{aligned}$$

for C as in (9). Therefore

$$\|\Pi(u_r, 1/2)\|_{L^\infty(B_1)} \leq \|\Pi(u_r, 1)\|_{L^\infty(B_1)} + \kappa_0$$

for $\kappa_0 = 2^{n+2}C$, which again can be iterated,

$$\begin{aligned}
& \|\Pi(u_r, 2^{-j})\|_{L^\infty(B_1)} \\
&= \|\Pi(u_{2^{-j+1}r}, 1/2)\|_{L^\infty(B_1)} \leq \|\Pi(u_{2^{-j+1}r}, 1)\|_{L^\infty(B_1)} + \kappa_0 \\
&= \|\Pi(u_{2^{-j+2}r}, 1/2)\|_{L^\infty(B_1)} + \kappa_0 \leq \|\Pi(u_{2^{-j+2}r}, 1)\|_{L^\infty(B_1)} + 2\kappa_0 \\
&\leq \dots \leq \|\Pi(u_r, 1)\|_{L^\infty(B_1)} + j\kappa_0.
\end{aligned}$$

By Lemma 3.5 with $K_0 > 2C$ as in (8),

$$\frac{1}{2} \|\Pi(u_r, 1)\|_{L^\infty(B_1)} \leq \|u_r\|_{L^\infty(B_1)} \leq 2 \|\Pi(u_r, 1)\|_{L^\infty(B_1)}. \quad (14)$$

Let $2^{-j-1}r \leq s \leq 2^{-j}r$, then

$$\begin{aligned}
\frac{\|u\|_{L^\infty(B_s)}}{s^2} &\geq \frac{\|u\|_{L^\infty(B_{2^{-j-1}r})}}{(2^{-j-1}r)^2} = \frac{1}{4} \|u_{2^{-j-1}r}\|_{L^\infty(B_1)} \\
&\geq \{(14)\} \geq \frac{1}{8} \|\Pi(u_r, 2^{-j-1})\|_{L^\infty(B_1)} \\
&\geq \{(10)\} \geq \frac{1}{8} (\|\Pi(u_r, 1)\|_{L^\infty(B_1)} + (j+1)\eta_0/2) \\
&\geq \{(14)\} \geq \frac{1}{16} \|u_r\|_{L^\infty(B_1)} + \frac{1}{16} (j+1)\eta_0 \\
&\geq \frac{1}{16} (\|u_r\|_{L^\infty(B_1)} + j\eta_0).
\end{aligned}$$

and

$$\begin{aligned}
\frac{\|u\|_{L^\infty(B_s)}}{s^2} &\leq 4 \frac{\|u\|_{L^\infty(B_{2^{-j}r})}}{(2^{-j}r)^2} = 4 \|u_{2^{-j}r}\|_{L^\infty(B_1)} \\
&\leq 8 \|\Pi(u_r, 2^{-j})\|_{L^\infty(B_1)} \leq 8 (\|\Pi(u_r, 1)\|_{L^\infty(B_1)} + j\kappa_0) \\
&\leq 16 \|u_r\|_{L^\infty(B_1)} + 8j\kappa_0 \\
&\leq 16 (\|u_r\|_{L^\infty(B_1)} + j\kappa_0).
\end{aligned}$$

where $\ln(\frac{r}{s}) - 1 \leq \ln(2)^{-1} \ln(\frac{r}{s}) - 1 \leq j \leq \ln(2)^{-1} \ln(\frac{r}{s}) \leq 2 \ln(\frac{r}{s})$ has been used. Finally, apply the proof to u/a if $a \neq 1$ to get (10) and (11). \square

The following lemma improves the estimate in Lemma 3.5, and says, combined with the previous lemma, that $|u_r - \Pi(u_r, 1)|$ decays like $(|\ln r|)^{-\alpha}$ at singular points.

Lemma 4.2. *Let u solve (3) in $B_1 \subset \mathbb{R}^n$ for $\|u\|_{L^\infty(B_1)} \leq M$ such that $u(0) = |\nabla u(0)| = 0$, $\sup_{0 < r \leq 1/4} |\psi_r| \leq C_\psi$, and f is Dini continuous with modulus of continuity ω_f and $f(0) = -a$, $a > 0$. Then if g^r solves*

$$\begin{aligned} \Delta g^r &= f(0)\chi_{\{\Pi(u_r, 1) > 0\}} - f(r\mathbf{x})\chi_{\{u_r > \psi_r\}} \quad \text{in } B_1, \\ g^r &= 0 \quad \text{on } \partial B_1, \end{aligned}$$

for $0 < r < c(\omega_f)$,

$$\|D^2 g^r\|_{L^2(B_1)} \leq C(M, n, \alpha, f, \psi)(\|\Pi(u_r, 1)\|_{L^\infty(B_1)})^{-\alpha}, \quad (15)$$

$$\|\Pi(g^r, s)\|_{L^\infty(B_1)} \leq C(M, n, \alpha, f, \psi, c)(\|\Pi(u_r, 1)\|_{L^\infty(B_1)})^{-\alpha}, \quad (16)$$

for $s \geq c > 0$.

Proof. From Lemma 3.5 and the boundedness of ψ_r ,

$$|u_r - \psi_r - \Pi(u_r, 1)| \leq |u_r - \Pi(u_r, 1)| + |\psi_r| \leq C(n, M, \psi, f)$$

so $\Pi(u_r, 1) < -C$ implies that $u_r - \psi_r < 0$, and $\Pi(u_r, 1) \geq C$ similarly gives that $u_r \geq \psi_r$. Therefore $|\Delta g^r| \leq |f(r\mathbf{x}) - f(0)| \leq \omega_f(r)$ outside the set $\{|\Pi(u_r, 1)| \leq C\}$ in B_1 . Now Lemma 3.4 concludes that

$$\begin{aligned} \|\Delta g^r\|_{L^2(B_1)}^2 &\leq |\{|\Pi(u_r, 1)| \leq C\}| + C\omega_f(r)^2 \\ &= \left| \left\{ \frac{|\Pi(u_r, 1)|}{\|\Pi(u_r, 1)\|_{L^\infty(B_1)}} \leq \frac{C}{\sup_{B_1} |\Pi(u_r, 1)|} \right\} \right| + C\omega_f(r)^2 \\ &\leq C \sup_{B_1} |\Pi(u_r, 1)|^{-\alpha} + C\omega_f(r)^2 \\ &\leq C(\|\Pi(u_r, 1)\|_{L^\infty(B_1)})^{-\alpha}, \quad \alpha \in (0, 1/4), r \leq c(\omega_f), \end{aligned}$$

where the last inequality is due to ω_f being Dini continuous while $(\|\Pi(u_r, 1)\|_{L^\infty(B_1)})^{-\alpha} \propto |\ln r|^{-\alpha}$ by Lemma 4.1 which is non-Dini. Finally L^p -theory (see e.g. Theorem 9.11 in [GT01]) implies (15).

Now let $\Pi(g^r, s) = \sum_{i,j=1}^n a_{ij} x_i x_j$. Then

$$\begin{aligned} \|D^2 \Pi(g^r, s)\|_{L^2(B_1)}^2 &= \int_{B_1} |D^2 \Pi(g^r, s)(\mathbf{x})|^2 d\mathbf{x} \leq \int_{B_1} |D^2 \frac{g^r(s\mathbf{x})}{s^2}|^2 dx \\ &= \frac{1}{s^n} \int_{B_s} |D^2 g^r(\mathbf{y})|^2 d\mathbf{y} \leq C \|D^2 g^r\|_{L^2(B_1)}^2 \\ &\leq \{(15)\} \leq C(\|\Pi(u_r, 1)\|_{L^\infty(B_1)})^{-\alpha} \end{aligned}$$

and

$$\|D^2\Pi(g^r, s)\|_{L^2(B_1)}^2 \geq \int_{B_1} \sum_{i,j=1}^n |a_{ij}| dx = C \sum_{i,j=1}^n |a_{ij}|$$

so (16) holds. \square

From this result, we can control how $\Pi(u, r)$ changes as r is halved. The proof goes through as in Corollary 7.3 in [ASW12] with [ASW12, Lemma 7.2] replaced by Lemma 4.2; the details are included for completeness.

Lemma 4.3. *Let u solve (3) in $B_1 \subset \mathbb{R}^n$ for $\|u\|_{L^\infty(B_1)} \leq M$ such that $u(0) = |\nabla u(0)| = 0$, $\sup_{0 < r \leq 1/4} |\psi_r| \leq C_\psi$, and f is Dini continuous with modulus of continuity ω_f and $f(0) = -a$, $a > 0$. Then*

$$\begin{aligned} & \|\Pi(u_r, 1/2) - \Pi(u_r, 1) - \Pi(Z_{\Pi(u_r, 1)}, 1/2)\|_{L^\infty(B_1)} \\ & \leq C(M, n, \alpha, \psi, f)(\|\Pi(u_r, 1)\|_{L^\infty(B_1)})^{-\alpha}. \end{aligned}$$

Proof. Write

$$u_r = \Pi(u_r, 1) + Z_{\Pi(u_r, 1)} + \tilde{g}^r + \tilde{h}^r,$$

where \tilde{g}^r and \tilde{h}^r have the properties

$$\begin{aligned} \Delta \tilde{g}^r &= \Delta(u_r - Z_{\Pi(u_r, 1)}), & \text{in } B_1 \\ \Delta \tilde{h}^r &= 0, & \text{in } B_1 \end{aligned}$$

and $\tilde{g}^r(0) = |\nabla \tilde{g}^r(0)| = \Pi(\tilde{g}^r, 1) = \tilde{h}^r(0) = |\nabla \tilde{h}^r(0)| = \Pi(\tilde{h}^r, 1) = 0$. Next, let g^r solve

$$\begin{aligned} \Delta g^r &= \Delta \tilde{g}^r & \text{in } B_1, \\ g^r &= 0 & \text{on } \partial B_1. \end{aligned}$$

Then $h^r := \tilde{g}^r + \tilde{h}^r - g^r$ is harmonic. By Lemma 4.2,

$$\|\Pi(g^r, 1/2)\|_{L^\infty(B_1)} \leq C(M, n, \alpha, \psi, f)(\|\Pi(u_r, 1)\|_{L^\infty(B_1)})^{-\alpha},$$

and

$$\begin{aligned} \|\Pi(h^r, 1/2)\|_{L^\infty(B_1)} &= \|\Pi(h^r, 1)\|_{L^\infty(B_1)} = \|\Pi(g^r, 1)\|_{L^\infty(B_1)} \\ &\leq C(M, n, \alpha, \psi, f)(\|\Pi(u_r, 1)\|_{L^\infty(B_1)})^{-\alpha} \end{aligned}$$

since h^r is harmonic and by the assumptions $\Pi(\tilde{g}^r, 1) = \Pi(\tilde{h}^r, 1) = 0$. Therefore

$$\begin{aligned} \Pi(u_r, 1/2) &= \Pi(\Pi(u_r, 1), 1/2) + \Pi(Z_{\Pi(u_r, 1)}, 1/2) + \Pi(g^r, 1/2) + \Pi(h^r, 1/2) \\ &= \Pi(u_r, 1) + \Pi(Z_{\Pi(u_r, 1)}, 1/2) + \Pi(g^r, 1/2) + \Pi(h^r, 1/2) \end{aligned}$$

by Lemma 3.1. We conclude,

$$\begin{aligned} & \|\Pi(u_r, 1/2) - \Pi(u_r, 1) - \Pi(Z_{\Pi(u_r, 1)}, 1/2)\|_{L^\infty(B_1)} \\ & \leq \|\Pi(g^r, 1/2)\|_{L^\infty(B_1)} + \|\Pi(h^r, 1/2)\|_{L^\infty(B_1)} \\ & \leq C(M, n, \alpha, \psi, f)(\|\Pi(u_r, 1)\|_{L^\infty(B_1)})^{-\alpha}. \end{aligned}$$

\square

5 Results

As mentioned in the introduction, the following theorems are the fruits of harvest from the preparation made in the earlier sections. The proofs follow the ones in [ASW12] with the following replacements: Lemma 3.5, Lemma 4.1, Lemma 4.3, Theorem 5.1, 5.2 and 5.4 replace [ASW12, Proposition 3.7, Corollary 5.3, 7.3, Theorem 8.1, 9.1 and 11.1] respectively. Note that [ASW12, Proposition 3.2, and Theorem 4.5] used in the proofs are valid for our problem as well. With this being said, the proofs of Theorem 5.1, 5.2 and 5.3 will nonetheless be included to demonstrate the techniques used. The main idea is to show monotonicity of the parameter $\delta_r := \delta(u_r)$ in the parametrization

$$\frac{\Pi(u_r, 1)}{\|\Pi(u_r, 1)\|_{L^\infty(B_1)}} = p_{\delta_r} = \pm[(1/2 + \delta_r)x^2 + (1/2 - \delta_r)y^2 - z^2]$$

as we go from r to $r/2$. This is done by using the fact that

$$\Pi(u_r, 1) - \Pi(u_r, 1/2) \approx \Pi(Z_{\Pi(u_r, 1)}, 1/2)$$

by Lemma 4.3 and exploiting the estimates of the coefficients of $A_x(\delta)$, $A_y(\delta)$, $A_z(\delta)$ and $A(\delta)$ given in Lemma (3.3).

The first result shows that, up to rotations, the blow-ups are contained in a discrete set. Recall the notation Z_p for solutions to (6)

Theorem 5.1. *Let u solve (3) in $B_1 \subset \mathbb{R}^3$ for $\|u\|_{L^\infty(B_1)} \leq M$ such that $u(0) = |\nabla u(0)| = 0$, $\sup_{0 < r \leq 1/4} |\psi_r| \leq C_\psi$, and f is Dini continuous with modulus of continuity ω_f and $f(0) = -a$, $a > 0$. Assume also that u is not $C^{1,1}$ at the origin, i.e., for any $K > 0$,*

$$\|u_r\|_{L^\infty(B_1)} \geq K$$

for $r \leq r_0(K)$. Then each limit of

$$\frac{u(r\mathbf{x})}{r^2} - \Pi(u_r, 1)$$

is contained in

$$\{Z_{p_1}(Q\mathbf{x}) : Q \in \mathcal{R}\} \cup \{Z_{p_2}(Q\mathbf{x}) : Q \in \mathcal{R}\} \cup \{Z_{p_3}(Qx) : Q \in \mathcal{R}\},$$

for $p_1 := \frac{x^2+y^2}{2} - z^2$, $p_2 := -p_1$ and $p_3 := x^2 - z^2$ respectively.

Proof. Assume otherwise and let u be a solution to (3) such that, after a rotation $Q \in \mathcal{R}$ and up to a sign, which we without loss of generality assume to be positive,

$$\lim_{j \rightarrow \infty} \frac{u_{r_j}}{\|u_{r_j}\|_{L^\infty(B_1)}} = p_{\delta_0}$$

for some $\delta_0 \in (0, 1/2)$, and where the parametrisation $p_\delta = (1/2 + \delta)x^2 + (1/2 - \delta)y^2 - z^2$ is used. Define δ_r so that, after a rotation $Q(r) \in \mathcal{R}$,

$$\frac{\Pi(u_r, 1)}{\|\Pi(u_r, 1)\|_{L^\infty(B_1)}} = p_{\delta_r},$$

and $\Pi(Z_{\Pi(u_r, 1)}, 1/2)$ by (7) can be written as

$$\frac{(3A_x(\delta_r) - A(\delta_r))x^2 + (3A_y(\delta_r) - A(\delta_r))z^2 + (3A_z(\delta_r) - A(\delta_r))y^2}{\|3x^2 - 1\|_{L^2(\partial B_1)}}.$$

We would like to write $\Pi(Z_{\Pi(u_r, 1)}, 1/2)$ in terms of p_{δ_r} : let $\kappa(\delta)$ be defined such that

$$1 - 2\delta + \kappa(\delta) = (1 + 2\delta) \frac{A_y(\delta) - A(\delta)}{A_x(\delta) - A(\delta)}.$$

Then

$$\begin{aligned} & \Pi(Z_{\Pi(u_r, 1)}, 1/2) \\ &= -\frac{3A_x(\delta_r) - A(\delta_r)}{\|3x^2 - 1\|_{L^2(\partial B_1)}(1 + 2\delta_r)} ((1 + 2\delta_r)x^2 + (1 + 2\delta_r) \frac{A_y(\delta_r) - A(\delta_r)}{A_x(\delta_r) - A(\delta_r)} y^2 \\ & \quad + (1 + 2\delta_r) \frac{A_z(\delta_r) - A(\delta_r)}{A_x(\delta_r) - A(\delta_r)} z^2) \\ &= -\frac{3A_x(\delta_r) - A(\delta_r)}{\|3x^2 - 1\|_{L^2(\partial B_1)}(1 + 2\delta_r)} ((1 + 2\delta_r)x^2 + (1 - 2\delta_r + \kappa(\delta_r))y^2 \\ & \quad + (1 + 2\delta_r) \frac{A_z(\delta_r) - A(\delta_r)}{A_x(\delta_r) - A(\delta_r)} z^2) \\ &= -\frac{3A_x(\delta_r) - A(\delta_r)}{\|3x^2 - 1\|_{L^2(\partial B_1)}(1 + 2\delta_r)} ((1 + 2\delta_r)x^2 + (1 - 2\delta_r + \kappa(\delta_r))y^2 \\ & \quad - (2 + \kappa(\delta_r))z^2), \\ &= -\frac{3A_x(\delta_r) - A(\delta_r)}{\|3x^2 - 1\|_{L^2(\partial B_1)}(1 + 2\delta_r)} (2p_{\delta_r} + \kappa(\delta_r)(y^2 - z^2)) \end{aligned}$$

where we used harmonicity in the second last equality. From Lemma 4.3 we therefore have

$$\begin{aligned} & |\Pi(u_r, 1/2) - \|\Pi(u_r, 1)\|_{L^\infty(B_1)} p_{\delta_r} \\ & \quad + \frac{3A_x(\delta_r) - A(\delta_r)}{\|3x^2 - 1\|_{L^2(\partial B_1)}(1 + 2\delta_r)} (2p_{\delta_r} + \kappa(\delta_r)(y^2 - z^2))| \\ & \leq C(M, \alpha, \psi, f)(\|\Pi(u_r, 1)\|_{L^\infty(B_1)})^{-\alpha}, \end{aligned} \tag{17}$$

from which $\Pi(u_r, 1/2)$ can be written as

$$\begin{aligned} & \left[\|\Pi(u_r, 1)\|_{L^\infty(B_1)} \left(\frac{1}{2} + \delta_r \right) - \frac{3A_x(\delta_r) - A(\delta_r)}{\|3x^2 - 1\|_{L^2(\partial B_1)}(1 + 2\delta_r)} (1 + 2\delta_r) \right] x^2 \\ & + \left[\|\Pi(u_r, 1)\|_{L^\infty(B_1)} \left(\frac{1}{2} - \delta_r \right) - \frac{3A_x(\delta_r) - A(\delta_r)}{\|3x^2 - 1\|_{L^2(\partial B_1)}(1 + 2\delta_r)} (1 - 2\delta_r + \kappa(\delta_r)) \right] y^2 \\ & - \left[\|\Pi(u_r, 1)\|_{L^\infty(B_1)} - \frac{3A_x(\delta_r) - A(\delta_r)}{\|3x^2 - 1\|_{L^2(\partial B_1)}(1 + 2\delta_r)} (2 + \kappa(\delta_r)) \right] z^2 \\ & + O((\|\Pi(u_r, 1)\|_{L^\infty(B_1)})^{-\alpha}). \end{aligned}$$

Note that $\Pi(u_r, 1/2)$ might have mixed terms with respect to the coordinate system chosen such that $\Pi(u_r, 1) = \|\Pi(u_r, 1)\|_{L^\infty(B_1)} p_{\delta_r}$. However, the coefficients in front of the mixed terms are of order $C(\sup_{B_1} |\Pi(u_r, 1)|)^{-\alpha}$ which can be seen by choosing points such that p_{δ_r} and $y^2 - z^2$ are zero. Therefore, by normalizing the z -coefficient, $1/2 + \delta_{r/2}$ is equal to

$$\begin{aligned} & \frac{\|\Pi(u_r, 1)\|_{L^\infty(B_1)}(1/2 + \delta_r) - \frac{3A_x(\delta_r) - A(\delta_r)}{\|3x^2 - 1\|_{L^2(\partial B_1)}(1 + 2\delta_r)} (1 + 2\delta_r)}{\|\Pi(u_r, 1)\|_{L^\infty(B_1)} - \frac{3A_x(\delta_r) - A(\delta_r)}{\|3x^2 - 1\|_{L^2(\partial B_1)}(1 + 2\delta_r)} (2 + \kappa(\delta_r))} \\ & + \frac{O((\|\Pi(u_r, 1)\|_{L^\infty(B_1)})^{-\alpha})}{|\ln r|} \\ & = 1/2 + \frac{\|\Pi(u_r, 1)\|_{L^\infty(B_1)} - 2 \frac{3A_x(\delta_r) - A(\delta_r)}{\|3x^2 - 1\|_{L^2(\partial B_1)}(1 + 2\delta_r)} \delta_r}{\|\Pi(u_r, 1)\|_{L^\infty(B_1)} - \frac{3A_x(\delta_r) - A(\delta_r)}{\|3x^2 - 1\|_{L^2(\partial B_1)}(1 + 2\delta_r)} (2 + \kappa(\delta_r))} \\ & + \frac{\frac{3A_x(\delta_r) - A(\delta_r)}{\|3x^2 - 1\|_{L^2(\partial B_1)}(1 + 2\delta_r)}}{\|\Pi(u_r, 1)\|_{L^\infty(B_1)} - \frac{3A_x(\delta_r) - A(\delta_r)}{\|3x^2 - 1\|_{L^2(\partial B_1)}(1 + 2\delta_r)} (2 + \kappa(\delta_r))} \kappa(\delta_r) \\ & + \frac{O((\|\Pi(u_r, 1)\|_{L^\infty(B_1)})^{-\alpha})}{|\ln r|} \end{aligned}$$

from which $\delta_{r/2}$ can be estimated,

$$\delta_{r/2} \leq \delta_r - C \frac{\kappa(\delta_r)}{\|\Pi(u_r, 1)\|_{L^\infty(B_1)}} + C_1 (\|\Pi(u_r, 1)\|_{L^\infty(B_1)})^{-1-\alpha} \quad (18)$$

$$\leq \delta_r - C \frac{\kappa(\delta_r)}{|\ln r|} + \frac{O((\|\Pi(u_r, 1)\|_{L^\infty(B_1)})^{-\alpha})}{|\ln r|} \quad (19)$$

where we used $\|\Pi(u_r, 1)\|_{L^\infty(B_1)} \propto |\ln r|$ by Lemma 4.2 and

$$-\infty < \frac{3A_x(\delta_r) - A(\delta_r)}{\|3x^2 - 1\|_{L^2(\partial B_1)}(1 + 2\delta_r)} \leq -C < 0.$$

Now, if $\delta_r \in [\beta, 1 - \beta]$, $\kappa(\delta) \geq c(\beta)$ by Lemma 3.3 which we together with $\|\Pi(u_r, 1)\|_{L^\infty(B_1)} \geq C$ apply in (19),

$$\delta_{r/2} \leq \delta_r - \frac{C}{|\ln r|}, \quad r \leq r_0(\beta, M, f)$$

hence

$$\delta_{2^{-k}r} \leq \delta_r - C \sum_{j=1}^{k-1} \frac{1}{j} \leq \delta_r - C \ln(k-1)$$

assuming that $\delta_{2^{-j}r} \in [\beta, 1-\beta]$ (note that $\delta_{2^{-j}r} \leq 1-\beta$ for every j if $\delta_r \leq 1-\beta$). From this estimate, we see that $\delta_{2^{-k}r} \leq c_0$ for c_0 in Lemma 3.3 eventually for k large depending on β , M , and f , which implies that $\kappa(\delta_{2^{-k}r}) \geq 2\delta_{2^{-k}r}$. Let us now consider two cases for a small constant $c = \min\{\alpha/2, C/2\kappa_0\}$ for C as in (18) and κ_0 in Lemma 4.1:

(i) $\delta_r \geq (\|\Pi(u_r, 1)\|_{L^\infty(B_1)})^{-c}$: Plugging this into (19),

$$\delta_{r/2} \leq \delta_r - \frac{C}{|\ln r|^{1+c}}$$

implying that $\delta_{2^{-j}r}$ is decreasing. If for some $j \geq 1$, $\delta_{2^{-j}r} < (\|\Pi(u_r, 2^{-j})\|_{L^\infty(B_1)})^{-\alpha/2}$ see case two. If not, then consider the limit of $\delta_{2^{-j}r}$ as $j \rightarrow \infty$. This limit has to coincide with $\delta_0 > 0$, but then we can iterate the argument above with β replaced by $\delta_0/2$, which in turn yields a logarithmic decay in j , contradicting that $\delta_{2^{-j}r} \rightarrow \delta_0$.

(ii) $\delta_r < (\|\Pi(u_r, 1)\|_{L^\infty(B_1)})^{-c}$: Then $\delta_{2^{-j}r} < (\|\Pi(u_r, 2^{-j})\|_{L^\infty(B_1)})^{-c}$ for all j ,

$$\begin{aligned} & \delta_{r/2} \\ & \leq \delta_r \left(1 - \frac{C}{\|\Pi(u_r, 1)\|_{L^\infty(B_1)}}\right) + C_1 (\|\Pi(u_r, 1)\|_{L^\infty(B_1)})^{-1-\alpha}, \\ & \leq \left(1 - \frac{C - C_1 (\|\Pi(u_r, 1)\|_{L^\infty(B_1)})^{-\alpha+c}}{\|\Pi(u_r, 1)\|_{L^\infty(B_1)}}\right) (\|\Pi(u_r, 1)\|_{L^\infty(B_1)})^{-c}, \end{aligned} \quad (20)$$

where we used the assumption $\delta_r < (\|\Pi(u_r, 1)\|_{L^\infty(B_1)})^{-c}$. Recall that $\|\Pi(u_r, 1/2)\|_{L^\infty(B_1)} \leq \|\Pi(u_r, 1)\|_{L^\infty(B_1)} + \kappa_0$ for κ_0 as in Lemma 4.1. This implies, by Taylor expansion, $\|\Pi(u_r, 1)\|_{L^\infty(B_1)} \geq M$, and c small, that

$$\begin{aligned} 1 - \frac{C - C_1 (\|\Pi(u_r, 1)\|_{L^\infty(B_1)})^{-\alpha+c}}{\|\Pi(u_r, 1)\|_{L^\infty(B_1)}} & \leq 1 - \frac{c\kappa}{\|\Pi(u_r, 1)\|_{L^\infty(B_1)}} \\ & \leq \left(\frac{\|\Pi(u_r, 1/2)\|_{L^\infty(B_1)}}{\|\Pi(u_r, 1)\|_{L^\infty(B_1)}}\right)^c \end{aligned}$$

which we in turn put into (20) to yield $\delta_{r/2} \leq \|\Pi(u_r, 1/2)\|_{L^\infty(B_1)}^{-c}$. This is iterated and consequently $\delta_{2^{-j}r} < (\|\Pi(u_r, 2^{-j})\|_{L^\infty(B_1)})^{-c}$ (note that the assumption $\|\Pi(u_r, 2^{-j})\|_{L^\infty(B_1)} \geq M$ holds due to $\|\Pi(u_r, 1/2)\|_{L^\infty(B_1)} \geq \|\Pi(u_r, 1)\|_{L^\infty(B_1)} + \eta_0/2$). Since $\|\Pi(u_r, 1)\|_{L^\infty(B_1)} \propto |\ln r|$, we deduce that $\delta_{2^{-k}r} \rightarrow 0$.

The conclusion is that if $\delta_r \in (0, 1/2)$, then $\delta_{2^{-k}r} \rightarrow 0$, as $k \rightarrow \infty$, contradicting that $\delta_{r_j} \rightarrow \delta_0 > 0$. \square

So far we have shown that δ_r converges to zero if there is a subsequence δ_{r_j} tending to zero, and even though it is insinuated by the proof above that the rotations $Q = Q(r)$ converges, this will be shown rigorously below with a quantitative estimate of the convergence speed. In particular, this shows uniqueness of blow-ups.

Theorem 5.2. *Let u solve (3) in $B_1 \subset \mathbb{R}^3$ for $\|u\|_{L^\infty(B_1)} \leq M$ such that $u(0) = |\nabla u(0)| = 0$, $\sup_{0 < r \leq 1/4} |\psi_r| \leq C_\psi$, and f is Dini continuous with modulus of continuity ω_f and $f(0) = -a$, $a > 0$. Assume also that for any $K > 0$,*

$$\|u_r\|_{L^\infty(B_1)} \geq K$$

for $r \leq r_0(K)$. Then there exists constants $R(M, \psi, f)$, $c(M, \psi, f)$ and $K(M, \psi, f)$ such that if

$$s \in (0, R), \quad \sup_{B_1} |\Pi(u, s)| \geq K, \quad \delta(u_s) \leq c,$$

then there is a rotation $Q \in \mathcal{R}$ such that

$$u_r - \Pi(u_r, 1) \rightarrow Z_{p_1}(Q \cdot) \tag{21}$$

or

$$u_r - \Pi(u_r, 1) \rightarrow Z_{p_2}(Q \cdot), \tag{22}$$

and

$$\left\| \frac{\Pi(u_r, 1)}{\|\Pi(u_r, 1)\|_{L^\infty(B_1)}} - \frac{p}{\|p\|_{L^\infty(B_1)}} \right\|_{L^\infty(B_1)} \leq C(M, \psi, f, \alpha) \left(K + \ln\left(\frac{s}{r}\right) \right)^{-c}$$

for all $r \in (0, s)$ with $p = p_1 := (x^2 + y^2)/2 - z^2$ in the case (21) and $p = p_2 := -p_1$ if (22) holds.

Proof. By (17) and the bound $\kappa(\delta) \leq 4\delta$,

$$\begin{aligned} \|\Pi(u_s, 2^{-k-1})\|_{L^\infty(B_1)} &= \|\Pi(u_s, 2^{-k})\|_{L^\infty(B_1)} + \frac{6A_x(\delta_{2^{-k}s}) - A(\delta_{2^{-k}s})}{\|3x^2 - 1\|_{L^2(\partial B_1)}(1 + 2\delta_{2^{-k}s})} \\ &\quad + O(\delta_{2^{-k}s}) + O(\Pi(u_s, 2^{-k})^{-\alpha}). \end{aligned}$$

If we also use $\tau_{2^{-k}s}$ to denote $\Pi(u_s, 2^{-k})\|_{L^\infty(B_1)} = \Pi(u_{2^{-k}s}, 1)\|_{L^\infty(B_1)}$ we infer

that, again by (17),

$$\begin{aligned}
& \left\| \frac{\Pi(u_s, 2^{-k})}{\|\Pi(u_s, 2^{-k})\|_{L^\infty(B_1)}} - \frac{\Pi(u_s, 2^{-k-1})}{\|\Pi(u_s, 2^{-k-1})\|_{L^\infty(B_1)}} \right\|_{L^\infty(B_1)} \\
& \leq \left\| \frac{\|\Pi(u_s, 2^{-k})\|_{L^\infty(B_1)} p_{\delta_r}}{\|\Pi(u_s, 2^{-k})\|_{L^\infty(B_1)}} \right. \\
& \quad - \frac{\tau_{2^{-k}s} p_{\delta_r} + \frac{3A_x(\delta_{2^{-k}s}) - A(\delta_{2^{-k}s})}{\|3x^2 - 1\|_{L^2(\partial B_1)}(1+2\delta_{2^{-k}s})} (2p_{\delta_{2^{-k}s}} + \kappa(\delta_{2^{-k}s})(y^2 - z^2))}{\tau_{2^{-k}s} + \frac{6A_x(\delta_{2^{-k}s}) - 2A(\delta_{2^{-k}s})}{\|3x^2 - 1\|_{L^2(\partial B_1)}(1+2\delta_{2^{-k}s})} + O(\delta_{2^{-k}s}) + O(\Pi(u_s, 2^{-k})^{-\alpha})} \\
& \quad \left. + C(M, \alpha, f, \psi)(\|\Pi(u_s, 2^{-k-1})\|_{L^\infty(B_1)})^{-1-\alpha} \right\|_{L^\infty(B_1)} \\
& \leq \left\| \frac{\tau_{2^{-k}s}(O(\delta_{2^{-k}s}) + O(\tau_{2^{-k}s}^{-\alpha}))}{\tau_{2^{-k}s}(\tau_{2^{-k}s} + \frac{6A_x(\delta_{2^{-k}s}) - 2A(\delta_{2^{-k}s})}{\|3x^2 - 1\|_{L^2(\partial B_1)}(1+2\delta_{2^{-k}s})} + O(\delta_{2^{-k}s}) + O(\Pi(u_s, 2^{-k})^{-\alpha}))} \right\|_{L^\infty(B_1)} \\
& \quad + C(M, \alpha, f, \psi)(\|\Pi(u_s, 2^{-k-1})\|_{L^\infty(B_1)})^{-1-\alpha} \\
& \leq C \frac{\delta_{2^{-k}s}}{\|\Pi(u_s, 2^{-k})\|_{L^\infty(B_1)}} + C(M, \alpha)(\|\Pi(u_s, 2^{-k-1})\|_{L^\infty(B_1)})^{-1-\alpha}.
\end{aligned}$$

Iteration of this inequality yields

$$\begin{aligned}
& \left\| \frac{\Pi(u_s, 2^{-k})}{\|\Pi(u_s, 2^{-k})\|_{L^\infty(B_1)}} - \frac{\Pi(u_s, 2^{-k-m})}{\|\Pi(u_s, 2^{-k-m})\|_{L^\infty(B_1)}} \right\|_{L^\infty(B_1)} \\
& \leq C \sum_{j=k}^{k+m} \frac{\delta_{2^{-j}s}}{\|\Pi(u_s, 2^{-j})\|_{L^\infty(B_1)}} + C(M, \alpha, f, \psi) \sum_{j=k}^{k+m} (\|\Pi(u_s, 2^{-j-1})\|_{L^\infty(B_1)})^{-1-\alpha}.
\end{aligned} \tag{23}$$

From the proof of the previous theorem, we can divide into the cases $\delta_{2^{-k}s} \geq (\|\Pi(u_s, 2^{-k})\|_{L^\infty(B_1)})^{-c}$ or $\delta_{2^{-k}s} \leq (\|\Pi(u_s, 2^{-k})\|_{L^\infty(B_1)})^{-c}$, and if the latter occurs for some k , then $\delta_{2^{-j}s} \leq (\|\Pi(u_s, 2^{-j})\|_{L^\infty(B_1)})^{-c}$ for all $j \geq k$, and it holds that such a k can be chosen only depending on M , f and ψ : similarly as in the previous proof,

$$\delta_{s/2} \leq \delta_s - C \frac{\delta_s}{\|\Pi(u_s, 1)\|_{L^\infty(B_1)}},$$

if $\delta_s \geq (\|\Pi(u_s, 1)\|_{L^\infty(B_1)})^{-c}$. Now if $\delta_{2^{-j}s} \geq (\|\Pi(u_s, 2^{-j})\|_{L^\infty(B_1)})^{-c}$ for $j < k_1$,

$$\delta_{2^{-k_1}s} \leq \delta_s \prod_{j=0}^{k_1-1} \left(1 - \frac{C}{\|\Pi(u_s, 2^{-j})\|_{L^\infty(B_1)}} \right).$$

From the Taylorexansion of $\ln(1-x)$ and Lemma 4.1,

$$\begin{aligned}
\ln \prod_{j=0}^{k_1-1} \left(1 - \frac{C}{\|\Pi(u_s, 2^{-j})\|_{L^\infty(B_1)}}\right) &= \sum_{j=0}^{k_1-1} \left(1 - \frac{C}{\|\Pi(u_s, 2^{-j})\|_{L^\infty(B_1)}}\right) \\
&\leq -\sum_{j=0}^{k_1-1} \frac{C}{2\|\Pi(u_s, 2^{-j})\|_{L^\infty(B_1)}} \\
&\leq -\sum_{j=0}^{k_1-1} \frac{C}{2(\|\Pi(u_s, 1)\|_{L^\infty(B_1)} + \kappa_0 j)} \\
&\leq -C/2 \ln \left(\frac{k_1 \kappa_0 + \|\Pi(u_s, 1)\|_{L^\infty(B_1)}}{\|\Pi(u_s, 1)\|_{L^\infty(B_1)}} \right)
\end{aligned}$$

so $\delta_{2^{-k_1}s} \leq \delta_s \left(\frac{\|\Pi(u_s, 1)\|_{L^\infty(B_1)}}{k_1 \kappa_0 + \|\Pi(u_s, 1)\|_{L^\infty(B_1)}} \right)^{C/2}$ and, if $2c \leq C/2$,

$$\begin{aligned}
\delta_{2^{-k_1}s} &\leq \delta_s \left(\frac{\|\Pi(u_s, 1)\|_{L^\infty(B_1)}}{k_1 \kappa_0 + \|\Pi(u_s, 1)\|_{L^\infty(B_1)}} \right)^{C/2} \\
&\leq \delta_s \left(\frac{\|\Pi(u_s, 1)\|_{L^\infty(B_1)}}{k_1 \kappa_0 + \|\Pi(u_s, 1)\|_{L^\infty(B_1)}} \right)^{2c} \\
&\leq \delta_s \left(\frac{\|\Pi(u_s, 1)\|_{L^\infty(B_1)}}{\|\Pi(u_s, 2^{-k_1})\|_{L^\infty(B_1)}} \right)^{2c},
\end{aligned}$$

where again Lemma 4.1 is used in the last inequality.

Let $k_1 \geq 0$ be the first number such that $\delta_{2^{-j}s} \leq (\|\Pi(u_s, 2^{-j})\|_{L^\infty(B_1)})^{-c}$. Then, by Lemma 4.3,

$$\|\Pi(u_s, 2^{-k_1+1})\|_{L^\infty(B_1)}^{-c} \leq \delta_{2^{-k_1+1}s} \leq \delta_s \left(\frac{\|\Pi(u_s, 1)\|_{L^\infty(B_1)}}{\|\Pi(u_s, 2^{-k_1+1})\|_{L^\infty(B_1)}} \right)^{2c}$$

from which

$$(k_1 - 1) \frac{\eta_0}{2} + \|\Pi(u_s, 1)\|_{L^\infty(B_1)} \leq \|\Pi(u_s, 2^{-k_1+1})\|_{L^\infty(B_1)} \leq \delta_s^{1/c} \|\Pi(u_s, 1)\|_{L^\infty(B_1)}^2$$

and

$$\begin{aligned}
k_1 &\leq \frac{2}{\eta_0} (\delta_s^{1/c} \|\Pi(u_s, 1)\|_{L^\infty(B_1)}^2 - \|\Pi(u_s, 1)\|_{L^\infty(B_1)}) + 1 \\
&\leq \frac{2}{\eta_0} \delta_s^{1/c} \|\Pi(u_s, 1)\|_{L^\infty(B_1)}^2.
\end{aligned}$$

Applying these results in (23) after letting $m \rightarrow \infty$,

$$\begin{aligned}
& \left\| \frac{\Pi(u_s, 2^{-k})}{\|\Pi(u_s, 2^{-k})\|_{L^\infty(B_1)}} - p \right\|_{L^\infty(B_1)} \\
& \leq C \delta_s^{1/c} \|\Pi(u_s, 1)\|_{L^\infty(B_1)}^2 \sum_{j=k}^{\infty} \|\Pi(u_s, 2^{-j})\|_{L^\infty(B_1)}^{-1-2c} \\
& \quad + C(M, \alpha, f, \psi) \sum_{j=k}^{\infty} (\|\Pi(u_s, 2^{-j-1})\|_{L^\infty(B_1)})^{-1-\alpha} \\
& \leq C \sum_{j=k}^{\infty} (\|\Pi(u_s, 2^{-k})\|_{L^\infty(B_1)} + ja\eta_0/2)^{-1-2c} \\
& \leq C \|\Pi(u_s, 2^{-k})\|_{L^\infty(B_1)}^{-2c} \\
& \leq C (\|\Pi(u_s, 1)\|_{L^\infty(B_1)} + ka\eta_0/2)^{-2c} \\
& \leq C \left(K(M) + \ln\left(\frac{s}{r}\right) \right)^{-2c},
\end{aligned}$$

for $r \in (2^{-k-1}s, 2^{-k}s)$. \square

By the uniqueness of limits $\lim_{j \rightarrow \infty} \frac{u_{r_j, \mathbf{x}^0}}{\|u_{r_j, \mathbf{x}^0}\|_{L^\infty(B_1)}}$ of the form

$$\pm \left(\frac{x^2 + y^2}{2} - z^2 \right),$$

we can characterize the free boundary $\partial\{u > \psi\}$ of solutions at singular points.

Theorem 5.3. *Let u solve (3) in $B_1 \subset \mathbb{R}^3$ for $\|u\|_{L^\infty(B_1)} \leq M$ such that $u(0) = |\nabla u(0)| = 0$, $\psi \in C^{1,\alpha}(B_r)$, $\sup_{0 < r \leq 1/4} |\psi_r| \leq C_\psi$, and f is Dini continuous with modulus of continuity ω_f and $f(0) = -a$, $a > 0$. Assume also that for any $K > 0$,*

$$\|u_r\|_{L^\infty(B_1)} \geq K$$

for $r \leq r_0(K)$, and that

$$\lim_{r \rightarrow 0} \frac{u_r}{\|u_r\|_{B_1}} = \pm p_1(Q \cdot)$$

for $p_1 = \frac{x^2 + y^2}{2} - z^2$ and $Q \in \mathcal{R}$. Then there are $r_0 = r_0(M, f, \psi)$ and Lipschitz functions g and h such that

$$\{u = \psi\} \cap B_R = \{(x, y, g(x, y))\} \cup \{(x, y, h(x, y))\} \cap B_{r_0}.$$

Also, $\sqrt{2}g - \sqrt{x^2 + y^2}$ and $\sqrt{2}h + \sqrt{x^2 + y^2}$ are C^1 functions.

Proof. By Theorem 5.2 we can without loss of generality assume the limit to be $p_1 := \frac{x^2 + y^2}{2} - z^2$. Since $\frac{u_r - \psi_r}{\|u_r\|_{L^\infty(B_1)}} =: v^r \rightarrow p_1$ in $C^{1,\alpha}(\overline{B_1})$, and $|\nabla p| \geq 1/2$ on $\overline{B_1} \setminus B_{1/2}$,

$$\{v^r = 0\} \cap (\overline{B_1} \setminus B_{1/2}) \subseteq \{\mathbf{x}' : \text{dist}(\mathbf{x}', x^2 + y^2 = 2z^2) < \sigma(r)\} \cap (\overline{B_1} \setminus B_{1/2}),$$

for some modulus of continuity σ , and consequently $\frac{\partial v^r}{\partial z} \leq -1/2$ on $\{u(r\mathbf{x}) = \psi(r\mathbf{x})\} \cap (\overline{B}_1 \setminus B_{1/2})$ for r small enough depending on σ . From the implicit function theorem and the $C^{1,\alpha}$ -regularity of u and ψ , $\{\mathbf{x} : u(r\mathbf{x}) = \psi(r\mathbf{x})\} \cap (\overline{B}_1 \setminus B_{1/2})$ can be expressed as a function $g^r(x, y)$ which is $C^{1,\alpha}(\overline{D}_1 \setminus D_{1/2})$ with $C^{1,\alpha}$ -norm independent of r , where $D_r = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < r^2\}$. Let g be the function given by gluing together such g^r , and define $g(0, 0) = 0$. Then g is Lipschitz in D_{r_0} and $g \in C^{1,\alpha}(\overline{D}_{r_0} \setminus D_s)$ for any $s > 0$, $\frac{g(rx, ry)}{r}$ is bounded in $C^{1,\alpha}(\overline{D}_1 \setminus D_{1/2})$ uniformly with respect to r and $\frac{g(rx, ry)}{r} \rightarrow \frac{\sqrt{x^2 + y^2}}{\sqrt{2}}$ in $C(\overline{D}_1 \setminus D_{1/2})$ from which it follows that $\frac{g(rx, ry)}{r} \rightarrow \frac{\sqrt{x^2 + y^2}}{\sqrt{2}}$ in $C^{1,\beta}(\overline{D}_1 \setminus D_{1/2})$, $\beta < \alpha$. Finally, for $(x, y) \rightarrow 0$, we infer that

$$\left| \nabla g(x, y) - \nabla \frac{\sqrt{x^2 + y^2}}{\sqrt{2}} \right| = \left| \nabla \frac{g(rx', ry')}{r} - \nabla \frac{\sqrt{x'^2 + y'^2}}{\sqrt{2}} \right| \rightarrow 0, \quad x', y' \in \partial D_1,$$

as $r \rightarrow 0$ and we conclude. \square

As mentioned, similar results are proven for limits $\lim_{j \rightarrow \infty} \frac{u_{r_j}}{\|u_{r_j}\|_{L^\infty(B_1)}}$ of the form $x^2 - z^2$, even though the analysis is somewhat more subtle due to the instability of the singularity [MW07].

Theorem 5.4 (Corresponds to Theorem 11.1 in [ASW12]). *Let u solve (3) in $B_1 \subset \mathbb{R}^3$ for $\|u\|_{L^\infty(B_1)} \leq M$ such that $u(0) = |\nabla u(0)| = 0$, $\psi \in C^{1,\alpha}(B_{r_\psi})$ for some $r_\psi > 0$, $\sup_{0 < r \leq 1/4} |\psi_r| \leq C_\psi$, and f is Dini continuous with modulus of continuity ω_f and $f(0) = -a$, $a > 0$. Assume also that for any $K > 0$,*

$$\|u_r\|_{L^\infty(B_1)} \geq K$$

for $r \leq r_0(K)$. If

$$\lim_{j \rightarrow \infty} u_{r_j} - \Pi(u_{r_j}, 1) = Z_{p_3}(Q \cdot)$$

for $p_3 = x^2 - z^2$, $Q \in \mathcal{R}$ and some sequence $\{r_j\}$, then

$$\lim_{r \rightarrow 0} u_r - \Pi(u_r, 1) = Z_{p_3}(Q \cdot).$$

Furthermore, for any $c > 0$, $\{u = \psi\} \cap K_c$ is contained two C^1 manifolds intersecting orthogonally, where $K_c = \{Q\mathbf{x} : y^2 < c(x^2 + z^2)\}$.

Finally we state the structural properties of the singular set S^u , which we practically split up as $S^u = S_1^u \cup S_2^u$ for

$$S_1^u := \left\{ \mathbf{x}^0 \in B_{1/2} : \lim_{r \rightarrow 0} \frac{u_{r, \mathbf{x}^0}(Q \cdot)}{\|u_{r, \mathbf{x}^0}\|_{L^\infty(B_1)}} \pm \left(\frac{x^2 + y^2}{2} - z^2 \right), Q \in \mathcal{R} \right\},$$

$$S_2^u := \left\{ \mathbf{x}^0 \in B_{1/2} : \lim_{r \rightarrow 0} \frac{u_{r, \mathbf{x}^0}(Q \cdot)}{\|u_{r, \mathbf{x}^0}\|_{L^\infty(B_1)}} x^2 - z^2, Q \in \mathcal{R} \right\}.$$

The results are that singular points in S_1^u are isolated while S_2^u is locally contained in a C^1 curve.

Theorem 5.5. *Let u solve (3) in $B_1 \subset \mathbb{R}^3$ for $\|u\|_{L^\infty(B_1)} \leq M$ such that $u(\mathbf{x}^0) = |\nabla u(\mathbf{x}^0)| = 0$, $\psi \in C^{1,\alpha}(B_{r_\psi})$ for some $r_\psi > 0$, $\sup_{0 < r \leq 1/4} |\psi_r| \leq C_\psi$, and f is Dini continuous with modulus of continuity ω_f and $f(\mathbf{x}^0) = -a$, $a > 0$. Then*

- (i) *if $x^0 \in S_1^u$, then x^0 is an isolated singular point*
- (ii) *if $x^0 \in S_2^u$, then $S_2^u \cap B_r(u)$ is contained in a C^1 curve.*

References

- [ASW10] John Andersson, Henrik Shahgholian, and Georg S. Weiss, *Uniform regularity close to cross singularities in an unstable free boundary problem*, Comm. Math. Phys. **296** (2010), no. 1, 251–270. MR 2606634 (2011c:35623)
- [ASW12] ———, *On the singularities of a free boundary through Fourier expansion*, Invent. Math. **187** (2012), no. 3, 535–587. MR 2891877
- [ASW13] ———, *The singular set of higher dimensional unstable obstacle type problems*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **24** (2013), no. 1, 123–146. MR 3034585
- [AW06] J. Andersson and G. S. Weiss, *Cross-shaped and degenerate singularities in an unstable elliptic free boundary problem*, J. Differential Equations **228** (2006), no. 2, 633–640. MR 2289547 (2007k:35522)
- [Caf77] Luis A. Caffarelli, *The regularity of free boundaries in higher dimensions*, Acta Math. **139** (1977), no. 3-4, 155–184. MR 0454350 (56 #12601)
- [Fre72] Jens Frehse, *On the regularity of the solution of a second order variational inequality*, Boll. Un. Mat. Ital. (4) **6** (1972), 312–315. MR 0318650 (47 #7197)
- [Gan01] M. I. Ganzburg, *Polynomial inequalities on measurable sets and their applications*, Constr. Approx. **17** (2001), no. 2, 275–306. MR 1814358 (2002d:26017)
- [GT01] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Springer, 2001.
- [KM96] Lavi Karp and Avmir S. Margulis, *Newtonian potential theory for unbounded sources and applications to free boundary problems*, J. Anal. Math. **70** (1996), 1–63. MR 1444257 (98c:35166)
- [MW07] R. Monneau and G. S. Weiss, *An unstable elliptic free boundary problem arising in solid combustion*, Duke Math. J. **136** (2007), no. 2, 321–341. MR 2286633 (2007k:35527)

- [Sch77] David G. Schaeffer, *Some examples of singularities in a free boundary*,
Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **4** (1977), no. 1, 133–144.
MR 0516201 (58 #24345)